On modules and rings in which complements are isomorphic to direct summands

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a joint work with

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Abstract

An important decomposition theorem in ring theory is the Wedderburn -Artin Theorem, which characterizes semisimple Artinian rings as a (unique) finite product of matrix rings over division rings. Recently, the authors of [1] state corresponding decomposition theorems for two more general classes of rings: right virtually semisimple rings and right completely virtually semisimple rings. These concepts are defined for modules, because of the fact that a right R-module M is semisimple if and only if every submodule of M is a direct summand. Thus, they called M *virtually semisimple* if every submodule of M is isomorphic to a direct summand of M. A non-zero indecomposable virtually semisimple left R-module is called virtually simple. On the other hand, M is called *completely virtually semisimple* if every submodule of M is virtually semisimple. A ring R is left (completely) virtually semisimple if the module R_R is (completely) virtually semisimple.

Let M be an R-module. The module M is called extending (or C1) if every complement submodule in M is a direct summand, or equivalently, if every submodule is essential in a direct summand. Extending modules and C2-modules (i.e., modules in which every submodule isomorphic to a direct summand, is a direct summand) are nice generalizations of injective modules. In this presentation, we introduce the notions of virtually extending (or CIS) and virtually C2-modules (VC2): An R-module M is called *virtually extending* if every complement submodule in M is isomorphic to a direct summand, and M is called a *virtually C2-module* if every complement submodule of M which is isomorphic to a direct summand, is itself a direct summand of M. Clearly, M is extending iff M is virtually extending and VC2. Among others, we will also discus that (1) if all cyclic sub-factors of a cyclic weakly co-Hopfian right R-module M (M is called *weakly co-Hopfian* if the image of every injective R-homomorphism on M is an essential submodule of M [5]) are virtually extending, then M is a finite direct sum of uniform submodules; (2) every distributive virtually extending module over any Noetherian ring is a direct sum of uniform submodules; (3) over a right Noetherian ring, every virtually extending module satisfies the Schröder-Bernstein property (M satisfies the *Schröder-Bernstein property* if whenever direct summands A and B of M are d-subisomorphic to each other, then $A \cong B$ [2]); (4) being virtually extending (VC2) is a Morita invariant property; (4) if $M \oplus E(M)$ is a VC2-module where $E(-)$ denotes the injective hull, then M is injective.

Keywords

Virtually semisimple module, virtually extending module, virtually C2 module, co-Hopfian module, square-free module, Osofsky-Smith Theorem, Schröder-Bernstein property.

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